Table of derivatives and integrals

Differentiation	Integration
$(cu)' = cu' \quad (c \text{ constant})$	$\int uv'\,dx = uv - \int u'v\,dx$ (by parts)
$(u + v)' = u' + v'$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$
$(uv)' = u'v + uv'$	$\int \frac{1}{x}\,dx = \ln \lvert x \rvert + c$
$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$	$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + c$
$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$ (Chain rule)	$\int \sin x\,dx = -\cos x + c$
	$\int \cos x\,dx = \sin x + c$
	$\int \tan x\,dx = -\ln \lvert \cos x \rvert + c$
$(x^n)' = nx^{n-1}$	$\int \cot x\,dx = \ln \lvert \sin x \rvert + c$
$(e^x)' = e^x$	$\int \sec x\,dx = \ln \lvert \sec x + \tan x \rvert + c$
$(e^{ax})' = ae^{ax}$	$\int \csc x\,dx = \ln \lvert \csc x - \cot x \rvert + c$
$(a^x)' = a^x \ln a$	$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$
$(\sin x)' = \cos x$	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$
$(\cos x)' = -\sin x$	$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{arcsinh} \frac{x}{a} + c$
$(\tan x)' = \sec^2 x$	$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{arccosh} \frac{x}{a} + c$
$(\cot x)' = -\csc^2 x$	$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$
$(\sinh x)' = \cosh x$	$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$
$(\cosh x)' = \sinh x$	$\int \tan^2 x\,dx = \tan x - x + c$
$(\ln x)' = \frac{1}{x}$	$\int \cot^2 x\,dx = -\cot x - x + c$
$(\log_a x)' = \frac{\log_a e}{x}$	$\int \ln x\,dx = x \ln x - x + c$
$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$	$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$
$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$	$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c$
$(\arctan x)' = \frac{1}{1 + x^2}$	
$(\operatorname{arccot} x)' = -\frac{1}{1 + x^2}$	

First order DEs

Separable equations

$$y' = f_1(x)f_2(y) \quad or \quad y' = \frac{f(x)}{g(y)}$$

The right-hand side must be a product of two functions, one of which depends upon *x* only whereas another one depends upon *y* only.

$$g(y)y' = f(x)$$
$$\Downarrow$$
$$g(y)\frac{dy}{dx} = f(x)$$
$$\Downarrow$$
$$g(y)dy = f(x)dx$$
$$\Downarrow$$
$$\int g(y)dy = \int f(x)dx + C$$

Such equations are solved by separation of variables. The idea is to collect all terms with *y* on one side and terms with *x* on another side by considering the differentials *dx* and *dy* as regular *x*- and *y*-terms, respectively.

Some special cases

Equation $y' = f\left(\frac{y}{x}\right)$ is transformed to separable with substitution $u = \frac{y}{x} \quad \Rightarrow \quad y = ux$

Equation $y' = f\left(ax + by + k\right)$ is transformed to separable with substitution $u = ax + by + k$

First order linear DEs

Equation $y'(x) + p(x)y(x) = r(x)$ is transformed to separable using the integrating factor $\rho(x) = e^{\int p(x)dx}$

Bernoulli equation $y'(x) + p(x)y(x) = r(x)[y(x)]^{\alpha} \quad (\alpha \neq 1 \& \alpha \neq 0)$ is transformed to linear using substitution $z(x) = [y(x)]^{1-\alpha}$

Second order linear differential equation with constant coefficients

$$y'' + ay' + by = r(x)$$
$$a, b - const.$$

Structure of general solution. General solution a sum of general solution of homogeneous equation and particular solution of the nonhomogeneous equation.

Homogeneous case

$$y'' + ay' + by = 0 \quad (1)$$

Solution is determined by solving the corresponding characteristic equations

$$\lambda^2 + a\lambda + b = 0 \quad (2)$$

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos\omega x$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

Particular solution of the non-homogeneous equation

$$y'' + ay' + by = r(x) \quad (4)$$
$$a, b - const.$$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Choice Rules for the Method of Undetermined Coefficients

(a) Basic Rule. *If $r(x)$ in (4) is one of the functions in the first column in Table 2.1, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into (4).*

(b) Modification Rule. *If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to (4), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).*

(c) Sum Rule. *If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.*

Systems of ODEs (planar case, critical/stationary points, stability)

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Assumed solution

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = const.$$

Eigen vector/value problem

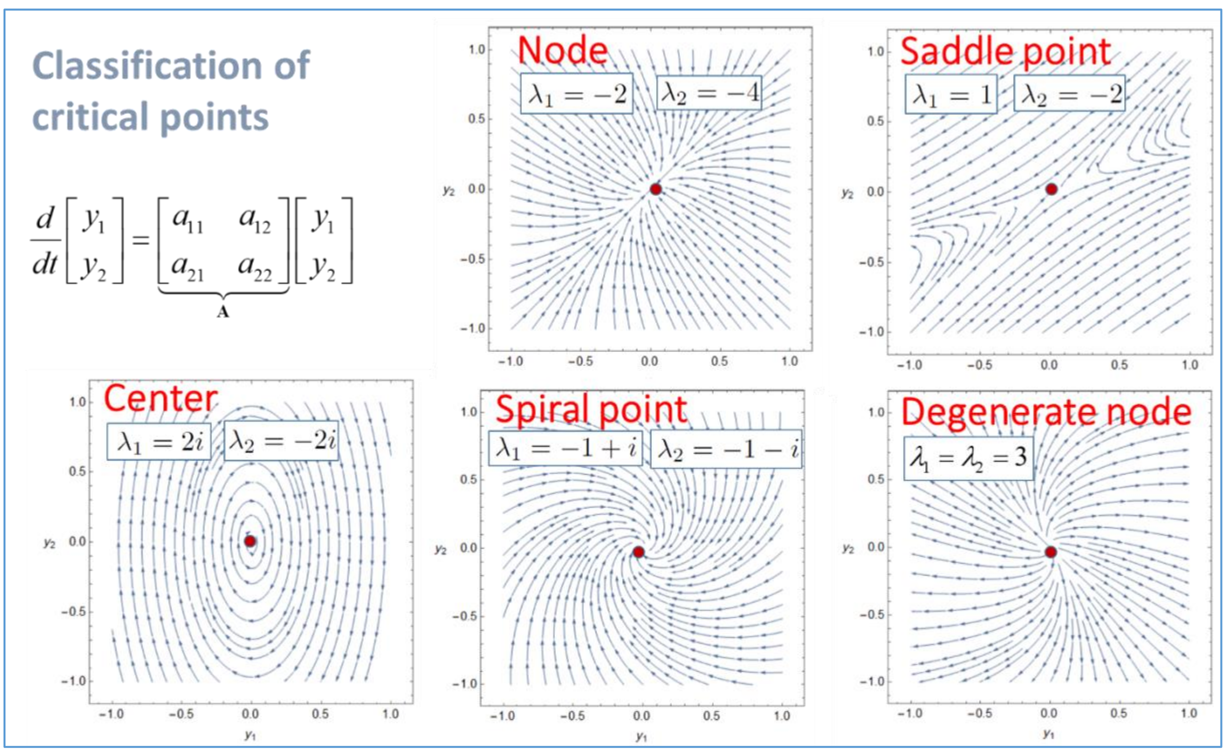
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic determinant
(condition of nonzero solution)

$$\det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = 0$$

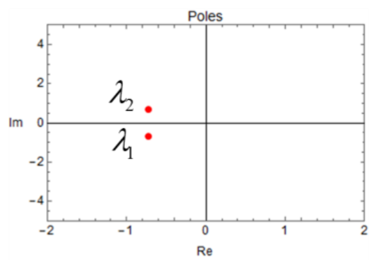
$$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$



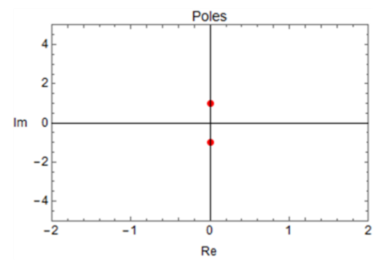
Stability concept

Practically stability properties are often determined after linearization by the location of eigen values (= roots of the characteristic equation) on the complex plane. The points showing eigen value locations are often called poles.

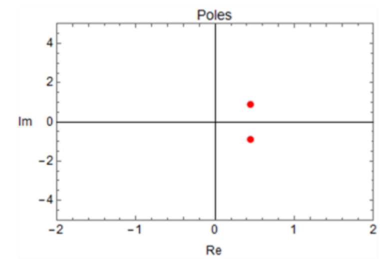
- If all the poles are on the left of the complex plane the system is asymptotically stable
- If at least one of the poles is on the imaginary axis the system is neutrally stable
- If at least one of the poles is on the right of the imaginary plane the system is unstable



asymptotically stable



neutrally stable



unstable

Laplace transform method

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Integration of Function

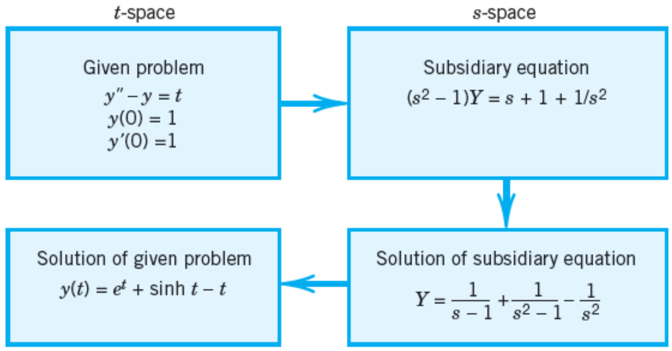
$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ $= \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$	t-Shifting (Second Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$	Differentiation of Transform
$\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \int_s^{\infty} F(\tilde{s})\,d\tilde{s}$	Integration of Transform
$\mathcal{L}(f) = \dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	f Periodic with Period p

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
1	$1/s$	1
2	$1/s^2$	t
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
7	$\dfrac{1}{s-a}$	e^{at}
8	$\dfrac{1}{(s-a)^2}$	te^{at}
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!}t^{n-1}e^{at}$
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}t^{k-1}e^{at}$

11	$\dfrac{1}{(s-a)(s-b)}$ $(a \neq b)$	$\dfrac{1}{a-b}(e^{at}-e^{bt})$
12	$\dfrac{s}{(s-a)(s-b)}$ $(a \neq b)$	$\dfrac{1}{a-b}(ae^{at}-be^{bt})$
13	$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\sin \omega t$
14	$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$
15	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$
16	$\dfrac{s}{s^2-a^2}$	$\cosh at$
17	$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}e^{at}\sinh \omega t$
18	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos \omega t$
19	$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1-\cos \omega t)$
20	$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}(\omega t-\sin \omega t)$

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
21	$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$
22	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\sin \omega t$
23	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$
24	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ $(a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$
25	$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$
26	$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2}\sin kt \sinh kt$
27	$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$
28	$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$

29	$\sqrt{s-a}-\sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2}I_0\left(\dfrac{a-b}{2}t\right)$
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
33	$\dfrac{1}{(s^2-a^2)^k} \quad (k>0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$
34	e^{-as}/s	$u(t-a)$
35	e^{-as}	$\delta(t-a)$
36	$\dfrac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$
37	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$
38	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$
39	$e^{-k\sqrt{s}} \quad (k>0)$	$\dfrac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$
40	$\dfrac{1}{s}\ln s$	$-\ln t-\gamma \quad (\gamma\approx 0.5772)$
41	$\ln\dfrac{s-a}{s-b}$	$\dfrac{1}{t}(e^{bt}-e^{at})$
42	$\ln\dfrac{s^2+\omega^2}{s^2}$	$\dfrac{2}{t}(1-\cos\omega t)$
43	$\ln\dfrac{s^2-a^2}{s^2}$	$\dfrac{2}{t}(1-\cosh at)$
44	$\arctan\dfrac{\omega}{s}$	$\dfrac{1}{t}\sin\omega t$
45	$\dfrac{1}{s}\operatorname{arccot} s$	$\mathrm{Si}(t)$



Linear systems of ODEs with Laplace transform

The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients

$$\text{(1)} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned}$$

Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(g_1)$, $G_2 = \mathcal{L}(g_2)$, we obtain from (1) the subsidiary system

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s). \end{aligned}$$

By collecting the Y_1- and Y_2-terms we have

$$\textbf{(2)} \qquad \begin{aligned} (a_{11} - s)Y_1 + \quad a_{12}Y_2 \quad &= -y_1(0) - G_1(s) \\ a_{21}Y_1 \quad + (a_{22} - s)Y_2 &= -y_2(0) - G_2(s). \end{aligned}$$

By solving this system algebraically for $Y_1(s), Y_2(s)$ and taking the inverse transform we obtain the solution $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$ of the given system (1).