SAMPLE PROBLEMS &SOLUTIONS FOR PQE IN MATH

Problem 1. *Heating an Office Building (Newton's Law of Cooling)*

Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. The outside temperature was about $T_A = 45°F$ all the night. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Solution. Step 1. Setting up a model. Experiments show that the time rate of change of the temperature *T* of a small ball-shaped body is proportional to the difference between *T* and the temperature of the surrounding medium (Newton's law of cooling):

$$\frac{dT}{dt} = k(T - T_A), \quad k < 0 \tag{1}$$

where T_A is the atmospheric temperature, and k is constant.

Such empirical laws like (1) are derived under idealized assumptions that rarely hold exactly. However, even if a model seems to fit the reality only poorly, it may still give valuable qualitative information. To see how good a model is, the engineer will collect experimental data and compare them with calculations from the model.

Step 2. General solution. The ODE (1) is separable. Separation, integration, and taking exponents gives the general solution:

$$\frac{dT}{dt} = k(T - T_A) \Rightarrow \frac{dT}{T - T_A} = kdt$$

$$\Rightarrow \ln|T - T_A| = kt + \ln C$$

$$\Rightarrow T - T_A = e^{kt + \ln C} = Ce^{kt}$$

Finally the general solution is

$$T(t) = T_A + Ce^{kt} \tag{2}$$

where C is an arbitrary constant.

Step 3. Particular solution. We choose 10 P.M. to be $t = 0$. Then the given initial condition is

$$T(0) = 70 \tag{3}$$

Substituting (2) in (3) gives $70 = 45 + C \Rightarrow C = 25$ and thus particular solution

$$T(t) = 45 + 25e^{kt} \tag{4}$$

where the parameter k is still unknown.

Step 4. Determination of k. We use $T(4)=65$ where $t=4$ corresponds to 2 A.M. Solving algebraically for *k* and inserting *k* into (4) gives

$$65 = 45 + 25e^{k4} \Rightarrow e^{k4} = (65-25)/25 = 4/5$$
$$k = (1/4)\ln(4/5) = -0.056$$

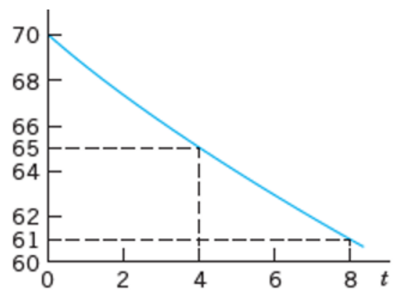
and

$$T(t) = 45 + 25e^{-0.056t} \tag{5}$$

Step 5. Answer and interpretation. 6 A.M. is 8 hours after 10 P.M., and

$$T(t) = 45 + 25e^{(-0.056)\cdot 8} = 61^{\circ}[F]$$

Hence the temperature in the building dropped 9°F. The entire temperature drop process is shown below:



Problem 2. Solve the initial value problem

$$\begin{aligned} &y'' + y = 0.001x^2 \\ &y(0) = 0 \\ &y'(0) = 1.5 \end{aligned} \tag{1}$$

From the mechanical standpoint, this is a *harmonic oscillator under the parabolic loading*, if *x* is interpreted as time and *y* is the displacement.

Solution. Step 1. General solution of the homogeneous ODE

$$y'' + y = 0$$

Assuming $y = e^{\lambda x}$ gives the characteristic equation $\lambda^2 + 1 = 0$ with a couple of complex conjugate (imaginary) roots $\lambda_{1,2} = \pm i$. In this case, according to the rule, the general solution of the homogeneous differential equation is

$$y_h = A\cos x + B\sin x \tag{2}$$

where *A* and *B* are arbitrary constants.

Step 2. Solution of the nonhomogeneous ODE. The method of undetermined coefficients suggests the following form of particular solution

$$y_p = K_2 x^2 + K_1 x + K_0 \tag{3}$$

Note that (3) includes the full quadratic polynomial even though the right hand side of equation (1) has the only one (quadratic) term. Now substituting (3) into equation (1) gives

$$2K_2 + K_2 x^2 + K_1 x + K_0 = 0.001x^2 \tag{4}$$

Comparing terms of the same degree of *x* on both sides of equation (4) gives the following set of linear algebraic equations

$$\begin{array}{l} x^0 \\ x^1 \\ x^2 \end{array} \quad \begin{cases} K_0 + 2K_2 = 0 \\ K_1 = 0 \\ K_2 = 0.001 \end{cases} \tag{5}$$

Substituting the solution of system (5) in (3) gives the particular solution of non-homogeneous equation (1) $y_p = 0.001x^2 - 0.002$ whose combination with (2) finally gives the general solution of non-homogeneous equation in the form

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002 \tag{6}$$

Step 3. Solution of the initial value problem. Substituting (6) into the initial conditions (1) gives

$$\begin{aligned} y(0) &= A\cos x + B\sin x + 0.001x^2 - 0.002\,|_{x=0} = A - 0.002 = 0 \\ y'(0) &= -A\sin x + B\cos x + 0.002x\,|_{x=0} = B = 1.5 \end{aligned}$$

Substituting A and B in (6) gives solution of the initial value problem (1)

$$y = y_h + y_p = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002 \tag{7}$$

Problem 3. Consider the following dynamical system on the plane $y_1 y_2$

$$\begin{aligned} \frac{dy_1(t)}{dt} &= -2y_1(t) + y_2(t) \\ \frac{dy_2(t)}{dt} &= y_1(t) - 2y_2(t) \end{aligned} \tag{1}$$

a) Represent system (1) in the matrix form;

b) Find eigen values of the system' main matrix and make a conclusion regarding stability or instability of the equilibrium (stationary) point of system (1) based on the roots of characteristic equation;

c) Using the Laplace transform method find solution of system (1) under the initial condition

$$\begin{aligned} y_1(0) &= 1 \\ y_2(0) &= 0 \end{aligned} \qquad (2)$$

Solution.

a) The matrix-column of unknowns and the matrix of coefficients are, respectively,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Then, system (1) takes the form

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} \qquad (3)$$

b) Represent the unknown vector in the form $\mathbf{y}(t) = \mathbf{x}e^{\lambda t}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a constant vector. Then equation (3) gives

$$\frac{d\mathbf{x}e^{\lambda t}}{dt} = \mathbf{A}\mathbf{x}e^{\lambda t} \Rightarrow \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$$

or the eigen vector problem

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{x} = \mathbf{0} \qquad (4)$$

where $\mathbf{I}$ is 2×2 identity matrix, and λ is called eigen value.

Equation (4) can have a non-zero solution $\mathbf{x}$ under condition

$$\begin{aligned} &\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0 \Rightarrow \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = 0 \Rightarrow (\lambda+2)^2 - 1 \\ &= \lambda^2 + 4\lambda + 3 = 0 \end{aligned} \qquad (5)$$

The characteristic equation (5) has two different negative real roots (eigen values of matrix **A**) $\lambda_1 = -3$ and $\lambda_2 = -1$. Geometrically such roots (poles) are located on the left half plane of the complex plane. Therefore, the equilibrium point is asymptotically stable (stable node).

c) Applying the Laplace transform to both sides of system (1) gives the algebraic system

$$\begin{aligned} sY_1 - y_1(0) &= -2Y_1 + Y_2 \\ sY_2 - y_2(0) &= Y_1 - 2Y_2 \end{aligned} \qquad (6)$$

where $Y_{1,2}$ are the so-called Laplace transform images.

Taking into account the initial conditions gives

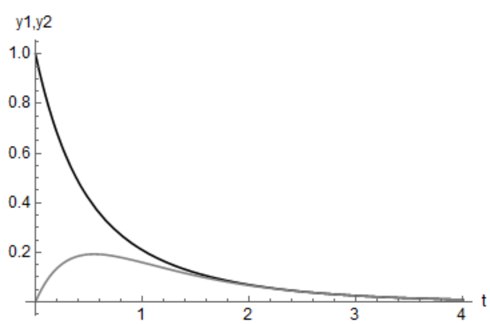
$$\begin{aligned} &-(2+s)Y_1 + Y_2 = 1 \\ &Y_1 - (2+s)Y_2 = 0 \end{aligned} \tag{7}$$

Solution of the linear algebraic system (7) can be found, for instance, by using the Kramer's rule as

$$\begin{aligned} Y_1 &= \frac{s+2}{s^2+4s+3} \\ Y_2 &= \frac{1}{s^2+4s+3} \end{aligned} \tag{8}$$

Note that the denominator coincides with the polynomial of the characteristic equation (5), therefore $s^2+4s+3=(s+1)(s+3)$ according to the basic property of polynomials. Now applying the inverse Laplace transform to (8) and using the table and the linearity property gives solution:

$$\begin{aligned} y_1 &= L^{-1}\left[\frac{s+2}{s^2+4s+3}\right] = L^{-1}\left[\frac{s+2}{(s+1)(s+3)}\right] \\ &= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right] = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \\ y_2 &= L^{-1}\left[\frac{1}{s^2+4s+3}\right] = L^{-1}\left[\frac{1}{(s+1)(s+3)}\right] \\ &= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right] = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{aligned} \tag{9}$$



Problem 4. *Vibration of a 2dof FE model of an elastic rod*
Consider the mechanical system as shown below in Fig.1. Assuming all the spring linear stiffness $k_i \ (i=1,2,3)$ and masse $m_i \ (i=1,2)$ are equal to unity, the differential equations of motion take the form of two coupled linear differential equations of the second order

$$
\begin{cases} \ddot{y}_1(t) + 2y_1(t) - y_2(t) = 0 \\ \ddot{y}_2(t) - y_1(t) + 2y_2(t) = 0 \end{cases} \tag{1}
$$

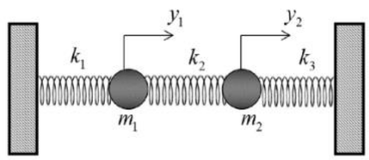


Figure 1 Two mass-spring finite element model of elastic rod

a) Represent system (1) in matrix form;

b) Determine two eigen frequencies of the model by solving the corresponding eigen-value problem for the stiffness matrix;

c) Applying the Laplace transform, find the particular solution of system (1) under the initial conditions

$$
\begin{aligned} & y_1(0) = 0, \quad y_1'(0) = 1 \\ & y_2(0) = 0, \quad y_2'(0) = 0 \end{aligned} \tag{2}
$$

Solution.

a) Follow the solution procedure of Problem 3. The matrix-column of unknowns and the matrix of coefficients are, respectively,

$$
\mathbf{y} = \begin{bmatrix} y_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
$$

Then, system (1) takes the form

$$
\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0 \tag{3}
$$

b) Represent the unknown vector in the form $\mathbf{y}(t) = \mathbf{x}e^{i\omega t}, \quad i^2 = -1$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a constant vector. Then equation (3) gives the eigen-vector problem

$$
\mathbf{Kx} = \lambda \mathbf{x} \Rightarrow (\mathbf{K} - \lambda \mathbf{I})\mathbf{x}, \quad \lambda = \omega^2 \tag{4}
$$

where $\mathbf{I}$ is 2×2 identity matrix, and $\lambda = \omega^2$ should be viewed as the eigen value.

Equation (4) can have a non-zero solution $\mathbf{x}$ under condition

$$
\begin{aligned} & \det(\mathbf{K} - \lambda \mathbf{I}) = 0 \Rightarrow \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 \\ & = \lambda^2 - 4\lambda + 3 = 0 \end{aligned} \tag{5}
$$

The roots are $\lambda_1 = \omega_1^2 = 1, \quad \lambda_2 = \omega_2^2 = 3$, therefore the eigen frequencies are $\omega_1 = 1, \quad \omega_2 = \sqrt{3}$.

c) Applying the Laplace transform to both sides of system (1) gives the algebraic system

$$
\begin{aligned}
&s^2Y_1 - \dot{y}_1(0) - sy_1(0) + 2Y_1 - Y_2 = 0 \\
&sY_2 - \dot{y}_2(0) - sy_2(0) - Y_1 + 2Y_2 = 0
\end{aligned} \tag{6}
$$

Taking into account the initial conditions (2) gives

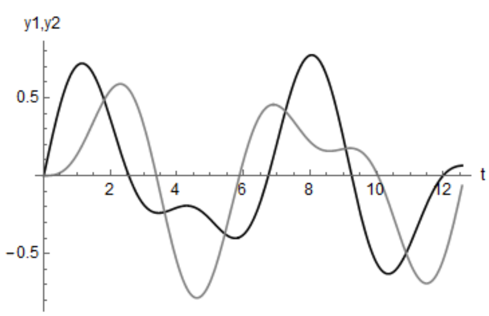
$$
\begin{aligned}
&(2+s^2)Y_1 - Y_2 = 1 \\
&-Y_1 + (2+s^2)Y_2 = 0
\end{aligned} \tag{7}
$$

Solution of the linear algebraic system (7) can be found, for instance, by using the Kramer's rule as

$$
\begin{aligned}
&Y_1 = \frac{s^2+2}{s^4+4s^2+3} \\
&Y_2 = \frac{1}{s^4+4s^2+3}
\end{aligned} \tag{8}
$$

Manipulating (8) in a similar to Problem 3 way, and applying the inverse Laplace transform gives

$$
\begin{aligned}
&y_1 = L^{-1}\left[\frac{s^2+2}{s^4+4s^2+3}\right] = L^{-1}\left[\frac{s^2+2}{(s^2+1)(s^2+3)}\right] \\
&= \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+3}\right] = \frac{\sin(t)}{2} + \frac{\sin\left(\sqrt{3}t\right)}{2\sqrt{3}} \\
&y_2 = L^{-1}\left[\frac{1}{s^4+4s^2+3}\right] = L^{-1}\left[\frac{1}{(s^2+1)(s^2+3)}\right] \\
&= \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s^2+3}\right] = \frac{\sin(t)}{2} - \frac{\sin\left(\sqrt{3}t\right)}{2\sqrt{3}}
\end{aligned} \tag{9}
$$



Problem 5. *Flow of water through a bottom hole of the liquid tank.*

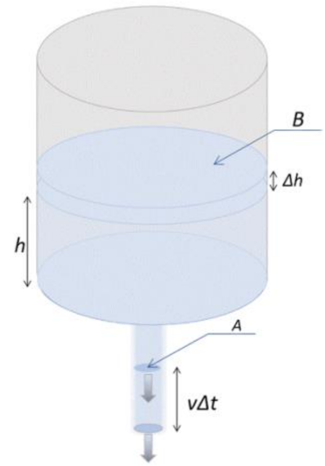


Figure 1 Flow of water through a hole of the tank

According to the Toricheli's law, the velocity of water through the hole depends upon the water height in the tank as $v = 0.6\sqrt{2gh}$, where the numerical factor 0.6 has empirical nature. Assuming the initial water height is $h(0) = h_0$, find a general expression for the time, which is needed to empty the tank. Calculate the time using the numbers in metric units:

$h_0 = 1.0\text{m}$

$A = 0.1\text{m}^2$

$B = 1.0\text{m}^2$

$g = 9.8\text{m/s}^2$

Solution. ***Step 1. Setting up a model.*** Using the mass conservation law with reference to Fig.1 gives $B\Delta h = -Av\Delta t$, where Δh is the water height drop during a small time increment Δt. Taking into account the Toricheli's law and considering an infinitely small temporal increment gives the differential equation

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}v = -\frac{A}{B}0.6\sqrt{2gh} \Rightarrow \frac{dh}{dt} = -\frac{A}{B}0.6\sqrt{2gh}$$

and thus the initial value problem

$$\frac{dh}{dt} = -k\sqrt{h}, \quad h(0) = h_0 \qquad \left(k = \frac{A}{B}0.6\sqrt{2g}\right) \tag{1}$$

Step 2.

Determining the general solution of the separable differential equation in (1)

$$\frac{dh}{dt} = -k\sqrt{h} \quad \Rightarrow \quad \frac{dh}{\sqrt{h}} = -kdt \quad \Rightarrow \quad 2\sqrt{h} = -kt + C \tag{2}$$

Step 3.

Satisfying the initial condition of the initial value problem (1)

$$t = 0: \quad 2\sqrt{h} = -kt + C \Rightarrow 2\sqrt{h_0} = C$$

$$2\sqrt{h} = -kt + 2\sqrt{h_0} \Rightarrow \sqrt{h} - \sqrt{h_0} = -\frac{1}{2}kt \tag{3}$$

Note that (3) is the particular integral, not a particular solution. In order to obtain the particular solution, equation (3) must be solved for *h*. However, the form (3) is sufficient for solving the problem completely.

Step 4. Finding the time, which is needed to empty the tank.

$$h = 0 : -\sqrt{h_0} = -\frac{1}{2} kt \Rightarrow t = \frac{2\sqrt{h_0}}{k}$$

Substituting k gives finally

$$t = \frac{1}{0.3\sqrt{2}} \frac{B}{A} \sqrt{\frac{h_0}{g}} = 2.35702 \frac{B}{A} \sqrt{\frac{h_0}{g}} \tag{4}$$

Using the given numbers gives

$$t = \frac{1}{0.3\sqrt{2}} \frac{1.0}{0.1} \sqrt{\frac{1.0}{9.8}} [\mathrm{s}] = 7.52923 [\mathrm{s}]$$