COURSE REVIEW

Particle Kinematics

Definitions

The absolute velocity and acceleration vectors of a particle or a material point P are defined, for each instant of time, with respect to a stationary, non-rotating observer: $\mathbf{v}_P = \frac{D\mathbf{r}_P}{Dt}$, $\mathbf{a}_P = \frac{D\mathbf{v}_P}{Dt}$, where $\frac{D}{Dt}$ is the time derivative w.r.t. a stationary observer. Note that velocity is always tangential to the particle's path (trajectory).

Basic kinematic relations for independent coordinates

$dv = a(t)dt,\ dv/a(v) = dt,\ vdv/a(v) = ds,\ vdv = a(s)ds$

Rectangular coordinates

$\mathbf{r}_P = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$, $\mathbf{v}_P = \dot{x}(t)\,\hat{\mathbf{i}} + \dot{y}(t)\,\hat{\mathbf{j}} + \dot{z}(t)\,\hat{\mathbf{k}}$

$\mathbf{a}_P = \ddot{x}(t)\,\hat{\mathbf{i}} + \ddot{y}(t)\,\hat{\mathbf{j}} + \ddot{z}(t)\,\hat{\mathbf{k}}$

where vectors with a cap denotes an unit directional vector.

Normal-tangential coordinates

$\mathbf{v} = v\,\hat{\mathbf{e}}_t = \dot{s}\,\hat{\mathbf{e}}_t = \rho\frac{d\theta}{dt}\,\hat{\mathbf{e}}_t$, where ρ = radius of curvature

Key derivative relation: $\frac{d\hat{\mathbf{e}}_t}{dt} = \frac{d\theta}{dt}\hat{\mathbf{e}}_n$

$\mathbf{a} = \dot{v}\,\hat{\mathbf{e}}_t + v\dot{\theta}\,\hat{\mathbf{e}}_n = \dot{v}\,\hat{\mathbf{e}}_t + v^2/\rho\,\hat{\mathbf{e}}_n$ (Osculating plane)

$\rho = [1 + (dy/dx)^2]^{3/2}/|d^2y/dx^2|$

Thus, acceleration is **never zero** for curvilinear motions.

Polar coordinates

$\mathbf{r} = r\,\hat{\mathbf{e}}_r,\ \mathbf{v} = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta,\ \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta$

Hints for solutions

1) Set up appropriate coordinate systems
2) Examine for kinematic constraints
   (Note: dot product may be very useful)
3) Apply vector form of kinematics equations

Rigid Body Kinematics

When does a rigid body rotate?

A rigid body rotates if at any instant, any line in the body changes its orientation with respect to a fixed reference line.

Define $\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt}$. For planar motions, $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, $\boldsymbol{\alpha} = \alpha\hat{\mathbf{k}}$ where ω = angular speed, α = angular acceleration

Body-fixed reference frames

This is a reference frame attached to the body. This frame rotates with the same angular velocity/acceleration as the rigid body.

Relative motion between two points of the same rigid body

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \boldsymbol{\omega}\times\mathbf{r}_{B/A}$, $v_{B/A} = r_{B/A}\,\omega$

- Relative velocity is orthogonal to the line joining A and B.
- Concept of instantaneous center of zero velocity (planar motions)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + \underbrace{\boldsymbol{\alpha}\times\mathbf{r}_{B/A}}_{\substack{\text{relative}\\ \text{tangential acc.}}} + \underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{B/A})}_{\substack{\text{relative}\\ \text{normal acceleration}}}$$

For planar motions, $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{B/A}) = -\omega^2\mathbf{r}_{B/A}$

1) For curvilinear translations, all material points of the rigid body have the same velocity and acceleration.
2) If a rigid body has a fixed point, then it cannot exhibit a curvilinear translation.

Rolling and slipping kinematics

Rolling without slip between two bodies in contact requires the velocities and tangential components of the accelerations of the points in contact to be equal. Slipping requires only the normal components of the velocities to be equal.

Relative motion analysis using rotating reference frames

This kinematic analysis is useful when A and B are material points of the same rigid body or of different rigid bodies.

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{\text{rel}} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A}, \text{ where } \mathbf{v}_{\text{rel}} = \left.\frac{d\mathbf{r}_{B/A}}{dt}\right|_{\text{frame at A}}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the frame attached at the point A. If the frame is attached to the rigid body which contains A, i.e., a body-fixed reference frame, then $\boldsymbol{\Omega} = \boldsymbol{\omega}$ (the angular velocity of that rigid body).

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{\text{rel}} + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \underbrace{\boldsymbol{\alpha} \times \mathbf{r}_{B/A}}_{\text{tangential acc.}} + \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A})}_{\text{normal acc.}}$$

where, $\boldsymbol{\alpha}$ is the angular acceleration of the rotating reference frame (or the rigid body if one uses a body-fixed frame) and $\mathbf{a}_{\text{rel}}$ is the relative acceleration measured by the rotating reference frame attached at A. The term $2\boldsymbol{\Omega} \times \mathbf{v}_{B/A}$ is the Coriolis acceleration, representing the difference in acceleration of B as measured by non-rotating and rotating reference frames.

Hints for solutions

1) Establish appropriate coordinate systems
2) Establish body-fixed reference frames, where appropriate
3) Determine $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ of the rigid body
4) Examine for kinematic constraints
5) Apply vector form of the kinematics equations

Particle Kinetics

Free Body Diagram

1) Include all external forces acting on the particle
2) Isolate the particle from kinematic constraints and replace constraints by constraint forces (reaction forces are normal to constraint surfaces)

Basic Equation of Motion (applicable at any instant of time)

$\sum \mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$ (A particle)

$\sum \mathbf{F} = m\frac{d\mathbf{v}_G}{dt} = m\mathbf{a}_G$ (A system of particles, G = center of mass)

Energy Approach (Scalar Approach)

Useful when problem gives information on change in position.
Work-Energy Principle: $\Delta T + \Delta V = U_{nc}$
(U_{nc} is the work done by non-conservative forces)
Kinetic energy $T = \frac{1}{2}mv^2$ (need to determine $\mathbf{v}$, then $v^2 = \mathbf{v} \cdot \mathbf{v}$)
Gravitational PE, $V_g = mgy$ (y is measured positive upward, need to choose datum)
Spring PE, $V_s = \frac{k}{2}(x - x_0)^2$, where x_0 is the unstretched position
Energy conservation: $U_{nc} = 0 \Rightarrow \Delta T + \Delta V = 0$, or $dE/dt = 0$

Momentum Principles

Useful when problem gives information on change in time.
Linear momentum principle:
$\int_{t_1}^{t_2} \sum \mathbf{F}\, dt = m\mathbf{v}_2 - m\mathbf{v}_1 \triangleq \mathbf{G}_2 - \mathbf{G}_1$ (for a particle)
where $\mathbf{G}$ is the linear momentum vector for the particle
$\sum_{j=1}^{N} \int_{t_1}^{t_2} \sum \mathbf{F}_j\, dt = m(\mathbf{v}_G)_{t_2} - m(\mathbf{v}_G)_{t_1}$ (for a system of particles)
Angular momentum principle:
$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}, \quad \int_{t_1}^{t_2} \sum \mathbf{M}_O\, dt = (\mathbf{H}_O)_{t_2} - (\mathbf{H}_O)_{t_1}$

Impact/Collision of particles (summary of equations):
Denote subscript n = direction of impact, t = tangential direction (perpendicular to the normal direction of impact)
1) $m_A v_{At} = m_A v'_{At}$ (prime denotes the instant right after impact)
2) $m_B v_{Bt} = m_B v'_{Bt}$
3) $m_A v_{An} + m_B v_{Bn} = m_A v'_{An} + m_B v'_{Bn}$
4) $e = -\frac{v'_{Bn} - v'_{An}}{v_{Bn} - v_{An}}$ (energy of system is conserved if $e = 1$)

Central-Force Problems: forces directed along $\hat{\mathbf{e}}_r$ direction, and angular momentum of system is conserved.

Motion in the vertical plane: under gravity force only, thus angular momentum is conserved in the vertical direction.

Hints for solutions

1) Establish appropriate coordinate systems
2) Draw free body diagrams
3) Examine for kinematic constraints
4) Identify which approach or combination of approaches to use – what is being asked for?
5) Apply vector form of kinematics equations
6) Apply appropriate kinetics equations

Rigid Body Kinetics

Free Body Diagrams

1) Include all external forces and body forces
2) Isolate the rigid body from kinematic constraints (e.g., rolling or slipping) and replace constraints by constraint forces

Basic Equation of Motion

$\sum \mathbf{F} = m\mathbf{a}_G$ (translation equations; G = center of mass)
$\sum M_G = I_G\alpha$ (rotation equation)
Above equations can be applied to the center of mass at each instant of time.

Alternate equations for the moment equation:
$\sum M_O = I_O\alpha$ (if O is a fixed point of the rigid body)
$\sum \mathbf{M}_A = \frac{d}{dt}\mathbf{H}_G + \mathbf{R}_G \times m\mathbf{a}_G$ (applied at an arbitrary point A)
$\sum M_A = I_G\alpha + (\mathbf{R}_G \times m\mathbf{a}_G)\cdot\hat{\mathbf{k}}$ (equation for planar motions)
Problem may involve more than one rigid body. Write the equations of motion for all bodies and solve simultaneously.

Mass Moment of Inertia

$I = \int r^2\, dm$ (r is measured from the point of reference)

- Mass moment of inertia for composite rigid bodies is obtained by adding the moments of inertia of all the individual bodies.

Parallel Axis Theorem: $I_A = I_G = md^2,\ d = \overline{AG}$
(Note: mass moment of inertia is smallest at the mass center)
Define radius of gyration: $k = \sqrt{I/m}$

***Energy Approach* (*Scalar Approach*)**

Useful when problem gives information on change in position
Work-Energy Principle: $\Delta T + \Delta V = U_{nc}$
Kinetic energy: $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$
Same energy conservation principle as used in particle kinetics. Problem may involve more than one rigid body. Find energies for each body and add them together to get total energy of the system, and then apply the work-energy principle.

Momentum Principles

Useful when problem gives information on change in time
$\int_{t_1}^{t_2} \sum \mathbf{F}\, dt = m(\mathbf{v}_G)_{t_2} - m(\mathbf{v}_1)_{t_1}$
$\int_{t_1}^{t_2} \sum \mathbf{M}_G\, dt = (\mathbf{H}_G)_{t_2} - (\mathbf{H}_G)_{t_1}, H_G = I_G\omega$ (or at fixed point O)
Apply the angular momentum equation at G (if convenient) or O (if available). Otherwise, apply the general equation at an arbitrary point A.

Hints for solutions

1) Establish appropriate body-fixed reference frames
2) Draw free body diagrams
3) Examine for kinematic constraints and establish appropriate kinematic relations
4) Identify which approach or combination of approaches to use – what is asked for?
5) Apply vector form of kinematics equations
6) Apply appropriate kinetics equations